14.7. Maximum and minimum values : local extrema

Def Consider a function $f(x,y)$.

(1) It has a critical point at (a,b) if it satisfies

$\nabla f(a,b) = \vec{0}$ or undefined.

not important for Math 215.

(2) It has a local maximum at (a,b) if it satisfies

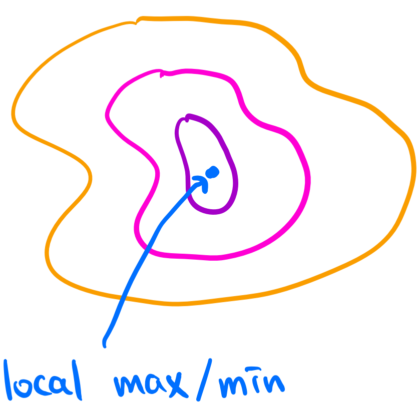
$f(x,y) \le f(a,b)$ near (a,b).

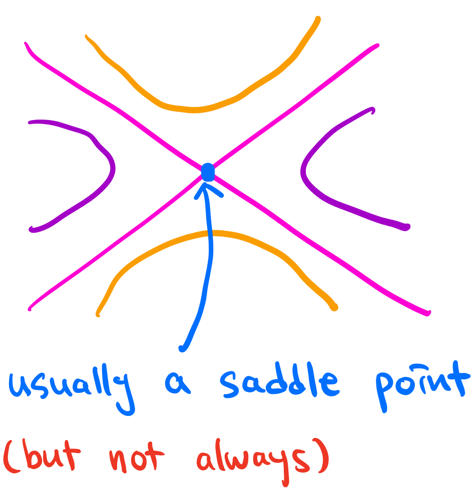
(3) It has a local minimum at (a,b) if it satisfies

$f(x,y) \ge f(a,b)$ near (a,b).

(4) It has a saddle point at (a,b) if it has a critical point at (a,b) which is not a local extremum.

Note On a contour map, critical points typically appear as follows:





Thm (Second derivative test)

Suppose that $f(x,y)$ is twice differentiable with $\nabla f(a,b) = \vec{0}$. Set

$$H := \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = f_{xx} \cdot f_{yy} - f_{xy}^2 .$$

(1) $H > 0$ and $f_{xx} > 0$: a local minimum at (a,b)

(2) $H > 0$ and $f_{xx} < 0$: a local maximum at (a,b)

(3) $H < 0$: a saddle point at (a,b).

Note (1) For $H = 0$, the test is inconclusive.

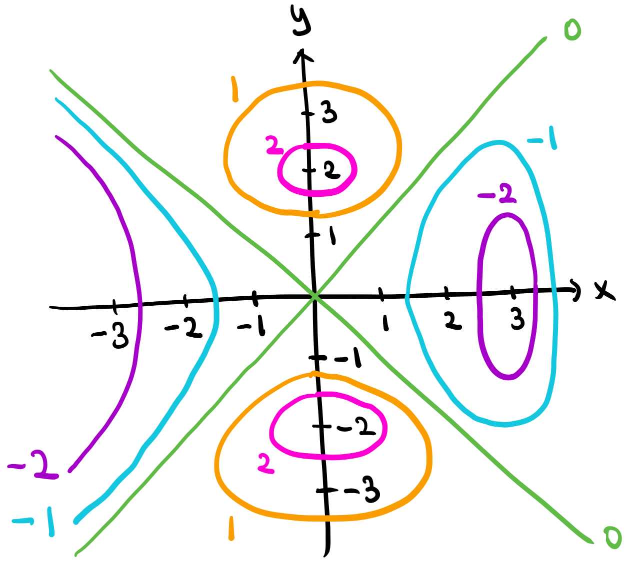
(2) For $H > 0$, you can use f_{yy} instead of f_{xx}.

(3) The quantity H (or the corresponding matrix) is called the Hessian of $f(x,y)$.

(4) There is a version of the first derivative test for multi-variable functions. However, it's practically useless.

(5) We will not consider the second derivative test for functions of three (or more) variables, as it requires some linear algebra.

Ex A contour map of $f(x,y)$ is given as follows:



Estimate the location of all local extrema and saddle points on the contour map.

Sol Levels decrease as you move toward $(3,0)$.

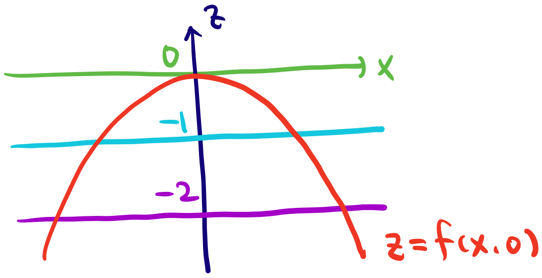
$\Rightarrow$ $\boxed{\text{A local minimum occurs at } (3,0)}$

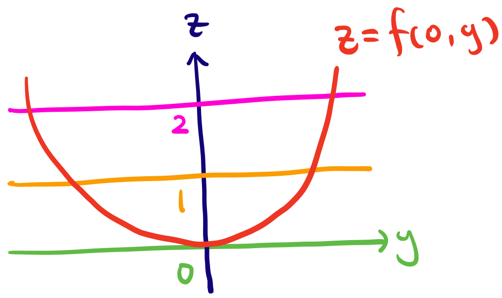
Levels increase as you move toward $(0,-2)$ or $(0,2)$.

$\Rightarrow$ $\boxed{\text{Local maxima occur at } (0,-2) \text{ and } (0,2)}$

Two level curves at level 0 intersect at $(0,0)$.

Here $f(x,y)$ attains a local maximum in one direction (along the x-axis) and a local minimum in another direction (along the y-axis)





$\Rightarrow$ $\boxed{\text{A saddle point occurs at } (0,0)}$

<u>Ex</u> Find and classify all critical points of the function

$$g(x,y) = e^{-y}(x^2+y^2).$$

<u>Sol</u> We first find all critical points.

$$g_x = \frac{\partial}{\partial x}\left(e^{-y}(x^2+y^2)\right) = e^{-y}\cdot 2x$$

$$g_y = \frac{\partial}{\partial y}\left(e^{-y}(x^2+y^2)\right) \underset{\substack{\uparrow \\ \text{product rule}}}{=} -e^{-y}(x^2+y^2) + e^{-y}\cdot 2y$$

$$= -e^{-y}(x^2+y^2-2y)$$

$$\nabla g = (g_x, g_y) = \left(2xe^{-y}, -e^{-y}(x^2+y^2-2y)\right)$$

At critical points, $\nabla g = (0,0)$

$$\begin{cases} 2x\underbrace{e^{-y}}_{\neq 0} = 0 \Rightarrow x = 0 \\ -\underbrace{e^{y}}_{\neq 0}(x^2+y^2-2y) = 0 \Rightarrow y^2-2y = 0 \Rightarrow y = 0 \text{ or } 2 \end{cases}$$

$\Rightarrow$ Critical points are at $(0,0)$ and $(0,2)$.

To classify these critical points, we apply the second derivative test.

The Hessian of $g(x,y)$ is

$$H = \det\begin{bmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{bmatrix} = g_{xx}\cdot g_{yy} - g_{xy}^2.$$

$$g_{xx} = \frac{\partial g_x}{\partial x} = \frac{\partial}{\partial x}(2xe^{-y}) = 2e^{-y}.$$

$$g_{xy} = \frac{\partial g_x}{\partial y} = \frac{\partial}{\partial y}(2xe^{-y}) = -2xe^{-y}.$$

$$g_{yy} = \frac{\partial g_y}{\partial y} = \frac{\partial}{\partial y}(-e^{-y}(x^2+y^2-2y))$$

$$= e^{-y}(x^2+y^2-2y) - e^{-y}(2y-2)$$

product rule

$$= e^{-y}(x^2+y^2-2y-2y+2)$$

$$= e^{-y}(x^2+y^2-4y+2)$$

At $(0,0)$: $H = \det\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 4, \quad f_{xx} = 2 > 0$

$\Rightarrow$ $\boxed{\text{A local minimum occurs at } (0,0)}$

At $(0,2)$: $H = \det\begin{bmatrix} 2e^{-2} & 0 \\ 0 & -2e^{-2} \end{bmatrix} < 0.$

$\Rightarrow$ $\boxed{\text{A saddle point occurs at } (0,2)}$